Available online at www.sciencedirect.com

ScienceDirect

International Journal of Heat and Mass Transfer 51 (2008) 576–585

International Journal of HEAT and MASS TRANSFER

www.elsevier.com/locate/ijhmt

Application of grey prediction to inverse nonlinear heat conduction problem

Jaw-Yeong Chiang, Cha'o-Kuang Chen *

Department of Mechanical Engineering, National Cheng Kung University, Tainan 70101, Taiwan, ROC

Received 24 September 2006; received in revised form 2 May 2007
Available online 27 July 2007

Abstract

The purpose of this research is to estimate the thermal conductivity with the inverse method which is modified by grey prediction; herein the thermal conductivity is a nonlinear function. When the thermal conductivity is the function of position and temperature, if one would try to obtain the thermal conductivity with the inverse method, then the measuring points of the temperature shall be distributed in whole object, consequently there would be a large number of measuring points for the relevant temperatures. The method of grey prediction will be able to dramatically decrease the number of measuring points for the temperature accordingly. However, the method of grey prediction should be accompanied with the prediction errors, thus the estimation of inverse method will produce a major deviation. This paper adopts the methods of the "rolling grey prediction" and the "comparison of temperature measurement" to correct the errors of grey prediction, and then proceed the inverse method to estimate the thermal conductivity. The estimated value obtained by the proposed method and the actual value compares very well.

Keywords: Grey prediction; Grey model; Rolling grey prediction; Inverse method; Prediction error; Measurement error

1. Introduction

In the modern industry, while it is related to the fields of designing and manufacturing, the acquirability for the thermal conductivity is indispensable. For example, the controllability of heat transfer in the designing of the IC and the manufacturing of the precision machinery, and the measurement of heat transfer for the composite and special material.

Inverse method used for estimating the thermal conductivity with single temperature parameter by means of the linear least-squares error method [1] had been reported in the literature [2], wherein it is shown that there is a potential to inverse thermal conductivity containing temperature parameter only by measuring the temperature at few measuring points. But in general, the thermal conductivity of an object is not only acting as a function of temperature, but also acting as a function of position as well. Therefore if one would like to inverse the thermal conductivity of an object via the method of temperature measurement, then it is necessary to measure all the temperatures along the entire length of such an object [3], and in this way, the measuring points for the temperature shall be increasing along the entire length of the object. Here, we will introduce the grey prediction method of grey system [4,5] into the previously stated inverse process and, thus it will be able to dramatically decrease the number of measuring points for the temperature so as to decrease both the measurement cost as well as the accumulation of the errors of measurement consequently.

In this paper, we introduce the principle of grey prediction first. Next, the Linear Matrix Equation by means of one-dimensional transient governing equation of the heat conduction is derived, and then illustrates the relevant method for estimating the thermal conductivity using the inverse method. Finally, the application of grey prediction

* Corresponding author. Tel.: +886 6 2757575; fax: +886 6 2342081.
E-mail address: ckchen@mail.ncku.edu.tw (C.-K. Chen).

0017-9310/$ - see front matter
doi:10.1016/j.ijheatmasstransfer.2007.05.015

Nomenclature

$\mathbf{A}$	coefficient matrix of vector $\mathbf{T}$
a	development coefficient
B	coefficient matrix of vector $[a \quad b]$
b	grey input
$\mathbf{C}$	vector constructed from the unknown K
$\mathbf{D}$	coefficient matrix of vector $\mathbf{C}$
$\mathbf{E}$	product of $\mathbf{A}^{-1}$ and $\mathbf{D}$
h	convection heat transfer coefficient
k	thermal conductivity
m	number of data sequence
N	number of the "measuring segment" points
N_s	initial point of the "measuring segment"
N_e	end point of the "measuring segment"
n	number of dispersed points
q_0	heat flux of the left side
q_n	heat flux of the right side
S	data sequence
$\mathbf{T}$	temperature vector
T	temperature
T_∞	temperature of the ambient
t	time
Δt	increment of time
x	parameter of grey prediction or spatial coordinate
Δx	increment of the parameter x
z	background value of grey model

Greek symbols

α	coefficient of z
η	number of data sequence
σ	standard deviation
ω	random variable
Φ, K	"quasi-mean thermal conductivity"
Θ	the temperature difference between the adjacent measure points

Subscripts

est	estimated data
exact	exact data
grey	grey prediction data
i	index of spatial coordinate
meas	measured data

Superscripts

(0)	original data of grey prediction
(1)	data proceeded by AGO once
j	index of time coordinate
$\wedge$	index of grey forecast

to this inverse method is elaborated. Herein, the method of grey prediction should be accompanied with the prediction for errors, therefore in this article, we try to correct it by utilizing the methods of the "rolling grey prediction" [6] and the "comparison of temperature measurement" so as to decrease the errors of grey prediction. Besides, since the occurrence of errors in the measurement is indispensable while performing the temperature measurement, in this paper, we have also presented the comparison on selecting the measuring points for temperatures in order to reduce the synergistic effect on the grey prediction with errors of measurement. The examples and conclusion disclosed in the last section of this article have identified the feasibility of this method.

2. Summary of grey prediction technique

It has been more than twenty years since the grey system theory proposed by a Chinese professor Julong Deng in the 1980s [4,5]. The analysis of grey prediction in the grey system is to explore the unknown large amount of information in a system containing incomplete data by utilizing the existing small amount of information [7]. This method establishes a grey model (GM) based on the known small amount of incomplete information, whereas the new data sequence generated from the known limited original data sequence via the accumulated generating operation (AGO) will have an obvious exponential pattern and, hence be able to eliminate the uncertainty of the original data sequence. Accordingly, it can establish a differential equation to perform the fitting work and, then to predict the unknown amount, to inversely acquire the predicted value of the original data sequence via the inverse accumulate generated operation (IAGO). The grey model GM(1,1) with a single parameter has been most widely applied, the detailed description is given in the following.

Considering a temperature sequence (take positive temperature or Kelvin temperature scale) with the number of items larger than or equal to 4 (at least 4 items) as the original data

$$S^{(0)} = \{T^{(0)}(1), T^{(0)}(2), \ldots, T^{(0)}(m)\}, \quad m = 1, 2, \ldots, \eta, \ \ \eta \geqslant 4 \tag{1}$$

wherein (0) represents original data of grey prediction. The data sequence proceeded by AGO once is

$$S^{(1)} = AGO \circ S^{(0)} = \{T^{(1)}(1), T^{(1)}(2), \ldots, T^{(1)}(m)\}, \quad m = 1, 2, \ldots, \eta, \ \ \eta \geqslant 4 \tag{2}$$

wherein

$$T^{(1)}(1) = T^{(0)}(1)$$
$$T^{(1)}(m) = T^{(1)}(m-1) + T^{(0)}(m), \quad m = 2, 3, \ldots, \eta, \ \ \eta \geqslant 4 \tag{3}$$

Eq. (2) indicates a monotonically increasing sequence (exponential pattern), which can fit a differential equation, i.e. the grey differential equation of the GM(1, 1) source model as

$$\frac{dT^{(1)}(x)}{dx} + aT^{(1)}(x) = b \tag{4}$$

wherein a is called the development coefficient and b is called the grey input, again

$$\frac{dT^{(1)}(x)}{dx} = \lim_{\Delta x \to 0} \frac{T^{(1)}(x+\Delta x) - T^{(1)}(x)}{\Delta x} \tag{5}$$

Δx represents the increment of the parameter x (x can be position, time or other usable parameter), and considered to be constant, so we can make it as the unit amount, while $T^{(1)}(x+\Delta x) - T^{(1)}(x)$ is data difference between the two former/latter points in the data sequence, therefore

$$\frac{dT^{(1)}(x)}{dx} \approx T^{(1)}(m) - T^{(1)}(m-1) = T^{(0)}(m), \quad m = 2, 3, \ldots, \eta, \ \eta \geqslant 4 \tag{6}$$

The definition for $T^{(1)}(x)$ is

$$T^{(1)}(x) \cong \alpha_m T^{(1)}(m) + (1-\alpha_m)T^{(1)}(m-1) = z^{(1)}(m), \quad m = 2, 3, \ldots, \eta, \ \eta \geqslant 4 \tag{7}$$

wherein $0 \leqslant \alpha_m \leqslant 1$. Usually it is taken as the mean value operation from the $T^{(1)}$ sequence, namely, $\alpha_m = 0.5$. $z^{(1)}(m)$ is termed background value, which can be combined with Eq. (3) and then converted into

$$z^{(1)}(m) = \alpha_m T^{(0)}(m) + T^{(1)}(m-1), \quad m = 2, 3, \ldots, \eta, \ \eta \geqslant 4 \tag{8}$$

While referring to Eqs. (6) and (7), Eq. (4) can be represented as

$$T^{(0)}(m) + az^{(1)}(m) = b, \quad m = 2, 3, \ldots, \eta, \ \eta \geqslant 4 \tag{9}$$

The above equation is the grey difference equation of GM(1, 1) model, However, the selection of α_m value will influence the accuracy of predicted value, hence it is necessary to have a correction [8], whereas this is the key point that this paper would like to look into.

With reference to Eq. (4) together with the initial condition $T^{(1)}(1) = T^{(0)}(1)$, the solution for Eq. (4) with discretization is

$$\hat{T}^{(1)}(m+1) = \left(T^{(0)}(1) - \frac{b}{a}\right)e^{-am} + \frac{b}{a}, \quad m \geqslant 0 \tag{10}$$

wherein ˆ represents Grey forecast and $\hat{T}^{(1)}(m+1)$ is the grey predicted value of the $S^{(1)}$. In addition, the result obtained after performing the IAGO operation based on Eq. (3)

$$\hat{T}^{(0)}(m+1) = \hat{T}^{(1)}(m+1) - \hat{T}^{(1)}(m), \quad m \geqslant 1 \tag{11}$$

The $\hat{T}^{(0)}(m+1)$ in the above equation, in fact, is the predicted value of the original temperature sequence $S^{(0)}$. It is found from the integration of both Eqs. (10) and (11), that the predicted value $\hat{T}^{(0)}(m+1)$ can be acquired as long as we can obtain the a and b values.From Eqs. (8) and (9), we can obtain a matrix equation

$$\begin{bmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -z^{(1)}(m) & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} T^{(0)}(2) \\ T^{(0)}(3) \\ \vdots \\ T^{(0)}(m) \end{bmatrix}, \quad m = 2, 3, \ldots, \eta, \ \eta \geqslant 4 \tag{12}$$

If the least square method is used, then the solution of a and b will be easily obtained

$$\begin{bmatrix} a \\ b \end{bmatrix} = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{T} \tag{13}$$

wherein

$$\mathbf{B} = \begin{bmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -z^{(1)}(m) & 1 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} T^{(0)}(2) \\ T^{(0)}(3) \\ \vdots \\ T^{(0)}(m) \end{bmatrix}, \quad m = 2, 3, \ldots, \eta, \ \eta \geqslant 4 \tag{14}$$

3. The derivation of physical model and inverse operation

With reference to Fig. 1, the basic assumption is as follows: (a) the ratio of the length to width of the target should be more than 10, its length is a supposed to be unit length; (b) the long perimeter of such target has been wrapped up by the insulated materials; (c) there is no any heat source inside such target; (d) its thermal conductivity contains the positional parameter.

The governing equation for the one-dimensional transient heat conduction without heat source is given by

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) \tag{15}$$

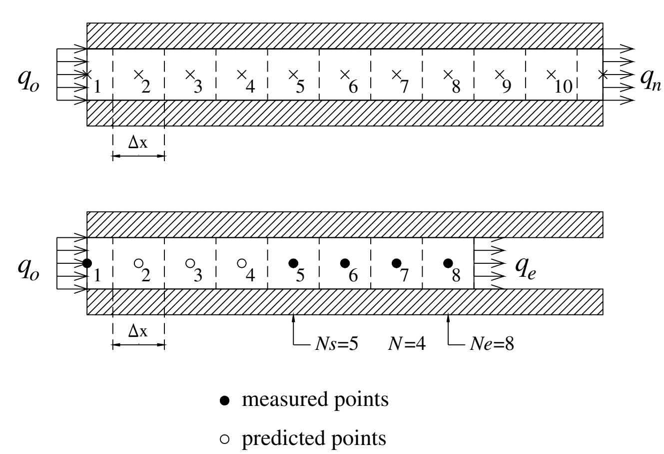

Fig. 1. The schematic view showing the related positions of the measured points and predicted points for the temperature.

Namely,

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + \frac{\partial k}{\partial x}\frac{\partial T}{\partial x} \tag{16}$$

T indicates the temperature, t indicates the time, x indicates the positional parameter, k indicates the thermal conductivity, wherein k contains the positional parameter. Here we add the product of density and specific heat into the thermal conductivity.

The discretization equation after being discretizated into $n-1$ units is

$$\frac{T_i^{j+1} - T_i^j}{\Delta t} = k_i^j \frac{T_{i+1}^j - 2T_i^j + T_{i-1}^j}{(\Delta x)^2} + \frac{k_{i+1}^j - k_{i-1}^j}{2\Delta x}\frac{T_{i+1}^j - T_{i-1}^j}{2\Delta x}, \quad 2 \leqslant i \leqslant n-1, \; j \geqslant 0 \tag{17}$$

Subscript i is the dispersed point of position x and superscript j is the dispersed point of time t. Eq. (17) can also be converted into

$$\frac{\Delta x}{\Delta t}(T_i^{j+1} - T_i^j) = \frac{(k_{i-1}^j + 4k_i^j - k_{i+1}^j)}{4}\frac{(T_{i-1}^j - T_i^j)}{\Delta x} - \frac{(k_{i+1}^j + 4k_i^j - k_{i-1}^j)}{4}\frac{(T_i^j - T_{i+1}^j)}{\Delta x}, \quad 2 \leqslant i \leqslant n-1, \; j \geqslant 0 \tag{18}$$

In the above equation, make

$$\Phi_i^j = \frac{(k_{i-1}^j + 4k_i^j - k_{i+1}^j)}{4}, \quad K_i^j = \frac{(k_{i+1}^j + 4k_i^j - k_{i-1}^j)}{4}, \quad 2 \leqslant i \leqslant n-1, \; j \geqslant 0 \tag{19}$$

Then Eq. (18) can be represented as

$$\frac{\Delta x}{\Delta t}(T_i^{j+1} - T_i^j) = \Phi_i^j \frac{(T_{i-1}^j - T_i^j)}{\Delta x} - K_i^j \frac{(T_i^j - T_{i+1}^j)}{\Delta x}, \quad 2 \leqslant i \leqslant n-1, \; j \geqslant 0 \tag{20}$$

wherein Φ_i^j and K_i^j are able to be called "quasi-mean thermal conductivity", Φ_i^j indicating the "quasi-mean thermal conductivity" between points of $i-1$ and i, and K_i^j indicating the "quasi-mean thermal conductivity" between points of i and $i+1$. From the relevant positions of each dispersed point, it can be known $\Phi_i^j = K_{i-1}^j$. It is found based on Eq. (19) and $\Phi_i^j = K_{i-1}^j$ that the thermal conductivity of each dispersed point is

$$k_i^j = \frac{(K_{i-1}^j + K_i^j)}{2}, \quad 2 \leqslant i \leqslant n-1, \; j \geqslant 0 \tag{21}$$

In this paper, it is assumable that the heat flux at the left side is a constant (q_0). According to the energy conservation law, when $i = 1$ (starting point at the left side) will be represented as the following equation:

$$\frac{\Delta x}{2}\frac{(T_1^{j+1} - T_1^j)}{\Delta t} = q_0 - K_1^j \frac{(T_1^j - T_2^j)}{\Delta x}, \quad j \geqslant 0 \tag{22}$$

The result from re-arrangement

$$\frac{\Delta x^2}{2\Delta t}(T_1^{j+1} - T_1^j) - q_0 \Delta x = K_1^j (T_2^j - T_1^j), \quad j \geqslant 0 \tag{23}$$

While $2 \leqslant i \leqslant n-1$ and $\Phi_i^j = K_{i-1}^j$, it is expressed as follows:

$$\frac{\Delta x^2}{\Delta t}(T_i^{j+1} - T_i^j) = K_{i-1}^j (T_{i-1}^j - T_i^j) - K_i^j (T_i^j - T_{i+1}^j), \quad 2 \leqslant i \leqslant n-1, \; j \geqslant 0 \tag{24}$$

again, according to the energy conservation law, while $i = n$ (end of right side), make the heat flux at the right boundary is q_n^j which is the function of time, and $\Phi_n^j = K_{n-1}^j$, then

$$\frac{\Delta x^2}{2\Delta t}(T_n^{j+1} - T_n^j) = K_{n-1}^j (T_{n-1}^j - T_n^j) - q_n^j \Delta x, \quad j \geqslant 0 \tag{25}$$

Based on Eqs. (23)–(25), it is applicable to develop a linear matrix equation as

$$\mathbf{AT} = \mathbf{DC} \tag{26}$$

wherein

$$\mathbf{A} = \begin{bmatrix} Q_0^j & \ddots & \ddots & & 0 \\ \ddots & \ddots & \ddots & 0 & \\ \ddots & 0 & \frac{\Delta x^2}{\Delta t} & 0 & \ddots \\ & 0 & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots & \frac{\Delta x^2}{2\Delta t} \end{bmatrix},$$

$$Q_0^j = \frac{\Delta x^2}{2\Delta t} - \frac{q_0 \Delta x}{(T_1^{j+1} - T_1^j)}, \quad j \geqslant 0 \tag{27}$$

$$\mathbf{T} = \begin{bmatrix} \cdots & (T_i^{j+1} - T_i^j) & \cdots \end{bmatrix}^{\mathrm{T}}, \quad 1 \leqslant i \leqslant n, \; j \geqslant 0 \tag{28}$$

$$\mathbf{D} = \begin{bmatrix} \ddots & \ddots & \ddots & & 0 \\ \ddots & \ddots & 0 & 0 & \\ \ddots & \ddots & \Theta_i^j & \ddots & \ddots \\ & 0 & -\Theta_i^j & \ddots & \ddots \\ 0 & & \ddots & \ddots & -\Delta x \end{bmatrix},$$

$$\Theta_i^j = (T_{i+1}^j - T_i^j), \quad 1 \leqslant i \leqslant n-1, \; j \geqslant 0 \tag{29}$$

$$\mathbf{C} = \begin{bmatrix} \cdots & K_i^j & \cdots & q_n^j \end{bmatrix}^{\mathrm{T}}, \quad 1 \leqslant i \leqslant n-1, \; j \geqslant 0 \tag{30}$$

Eq. (26) will be rewritten as

$$\mathbf{T} = \mathbf{A}^{-1}\mathbf{DC} = \mathbf{EC}, \quad \mathbf{E} = \mathbf{A}^{-1}\mathbf{D} \tag{31}$$

Here it is found via the linear least-squares error method of the inverse method

$$\mathbf{C}_{\mathrm{est}} = (\mathbf{E}^{\mathrm{T}}\mathbf{E})^{-1}\mathbf{E}^{\mathrm{T}}\mathbf{T}_{\mathrm{meas}} \tag{32}$$

wherein $\mathbf{T}_{\text{meas}}$ is the direct measured temperature, and $\mathbf{C}_{\text{est}}$ is the estimation resulted from the quasi-mean thermal conductivity K_i^j together with the heat flux at the right side q_n^j. Then the acquired K_i^j value is substituted back into Eqs. (21) and (19) so as to obtain the thermal conductivity k_i^j. But the thermal conductivity k is a function of position, if one would like to acquire the k using the inverse method according to Eq. (32), it is indispensable to measure the temperature at each dispersed point. Therefore, this method will need a large number of measuring points. In this article, we are proposing the concept of grey prediction for exploration in order to find out a better method.

4. The applications of grey prediction

With reference to Fig. 1, there are two areas that the grey prediction proposed in this study can be applied to, the first area is the grey prediction on temperature at the front segment of the target after temperature measurement operation, wherein the measured temperature acts as the original data sequence of grey prediction so as to carry on the prediction of temperature at the unmeasured points positioned in the front segment of such target. Another area is the grey prediction on the quasi-mean thermal conductivity K of the latter segment of the target based on the quasi-mean thermal conductivity K of the front segment of such target, which is estimated by the inverse method. Both perform the equidistant grey prediction at the positional point.

In this method, the temperature measurement at the first measuring point ($i = 1$) of the target is adopted as the reference temperature, and then the temperature of N successive points (also referred to as "measuring segment") will be measured and taken as the original data sequence for the grey prediction in order to perform the "rolling grey prediction" of the temperature along the decreasing direction of x. In this way, it will be able to acquire all the temperature starting from the first point ($i = 1$) of the target to the "measuring segment". If N_s is the initial point of the "measuring segment", and N_e is the end point of the "measuring segment", namely we can get the temperature from $i = 1$ to $i = N_e$.

The so-called "rolling grey prediction" is to take the "measuring segment" temperature (the segment temperature from N_s to N_e) as the original data sequence of grey prediction to predict the temperature of the next point through grey prediction (since it is utilizing x along the decreasing direction in this paper, the next point indicates the $N_s - 1$ point), and again adopting temperature from the $N_e - 1$ point to the $N_s - 1$ point as the original data sequence in order to perform the prediction of temperature at the next point (the $N_s - 2$ point). Apply the same rule and continue predicting all the way to the first point ($i = 1$) of the target. The values acquired by the grey prediction, however, will accompany with some inaccuracy, and the accumulated inaccuracy cannot be underestimated. This situation needs to be proved through the background value.

From the background value of Eq. (8), we know that the selection of α_m value will affect the predicted accuracy. The countermeasure in this study is to use the first point ($i = 1$) temperature of the target from the rolling grey prediction and compare it with the directly measured temperature at the same point ($i = 1$). If it does not fit in the set error range (here is set as ±0.1 °C), repeat the stated rolling grey prediction until it meets the requirement in order to decide the α_m value. During the repetition process, start from $\alpha_m = 0.0$, and add an increment (here is set at 1×10^{-5}) when each time to repeat the rolling grey prediction until $\alpha_m = 1.0$.

In this paper the whole target will be dispersed into 10 units (i.e. $n = 11$), and adopt $N = 4$, then $1 \leqslant N_s \leqslant 8$ ($1 \leqslant N_s \leqslant (n - N + 1)$). According to Eq. (10) can be made $m = N = 4$, thus

$$\hat{T}^{(1)}(5) = \left(T^{(0)}(1) - \frac{b}{a}\right)e^{-4a} + \frac{b}{a} \tag{33}$$

wherein a and b will be obtained from Eq. (13), namely $\begin{bmatrix} a \\ b \end{bmatrix} = (\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{T}$, based on Eqs. (8) and (9) can be known

$$\mathbf{B} = \begin{bmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ -z^{(1)}(4) & 1 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} T^{(0)}(2) \\ T^{(0)}(3) \\ T^{(0)}(4) \end{bmatrix} \tag{34}$$

The predicted value of the measured temperatures sequence ($\hat{T}^{(0)}(5)$) will be gained after performing the IAGO operation based on Eq. (3), namely $\hat{T}^{(0)}(5) = \hat{T}^{(1)}(5) - T^{(1)}(4)$. Since it is utilizing x along the decreasing direction, the next point indicates the $N_s - 1$ point, i.e. $\hat{T}^{(0)}(5) = \hat{T}^{(0)}(N_s - 1)$. Using the same method, we can get $\hat{T}^{(0)}(6), \hat{T}^{(0)}(7), \hat{T}^{(0)}(8), \ldots,$ successively, namely we can get $\hat{T}^{(0)}(N_s - 2), \hat{T}^{(0)}(N_s - 3), \hat{T}^{(0)}(N_s - 4), \ldots, \hat{T}^{(0)}(1)$. In other words, the temperature can be found from $i = 1$ to $i = N_e$, i.e. $\hat{T}^{(0)}(1), \hat{T}^{(0)}(2), \ldots, T^{(0)}(N_s), \ldots, T^{(0)}(N_e)$. Also, this temperature sequence can be written as $T_1^j, T_2^j, \ldots, T_{Ns}^j, \ldots, T_{Ne}^j$ too.

While after using the grey prediction method to obtain the temperature sequence $T_1^j, T_2^j, \ldots, T_{Ns}^j, \ldots, T_{Ne}^j$, a linear matrix equation similar to Eq. (26) could be consisted in accordance with Eqs. (23)–(25), wherein n is replaced by N_e, and also the q_n^j will be replaced by the heat flux q_e^j at the right side of the N_e point, namely we can derive a linear matrix equation again, as follows:

$$\mathbf{AT} = \mathbf{DC} \tag{35}$$

wherein

$$\mathbf{A} = \begin{bmatrix} Q_0^j & \ddots & \ddots & & 0 \\ \ddots & \ddots & \ddots & 0 & \\ \ddots & 0 & \frac{\Delta x^2}{\Delta t} & 0 & \ddots \\ & 0 & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix},$$

$$Q_0^j = \frac{\Delta x^2}{2\Delta t} - \frac{q_0 \Delta x}{(T_1^{j+1} - T_1^j)}, \quad j \geqslant 0 \tag{36}$$

$$\mathbf{T} = \begin{bmatrix} \cdots & (T_i^{j+1} - T_i^j) & \cdots \end{bmatrix}^{\mathrm{T}}, \quad 1 \leqslant i \leqslant N_{\mathrm{e}}, \ j \geqslant 0 \tag{37}$$

$$\mathbf{D} = \begin{bmatrix} \ddots & \ddots & \ddots & & 0 \\ \ddots & \ddots & 0 & 0 & \\ \ddots & \ddots & \Theta_i^j & \ddots & \ddots \\ & 0 & -\Theta_i^j & \ddots & \ddots \\ 0 & & \ddots & \ddots & -\Delta x \end{bmatrix},$$

$$\Theta_i^j = (T_{i+1}^j - T_i^j), 1 \leqslant i \leqslant N_{\mathrm{e}} - 1, \ j \geqslant 0 \tag{38}$$

$$\mathbf{C} = \begin{bmatrix} \cdots & K_i^j & \cdots & q_{\mathrm{e}}^j \end{bmatrix}^{\mathrm{T}}, \quad 1 \leqslant i \leqslant N_{\mathrm{e}} - 1, \ j \geqslant 0 \tag{39}$$

The representative meaning of Eq. (35) is similar to Eq. (26), and the only difference is that the used points related to such equation are from the first point to the N_{e} point of the target instead, so there are N_{e} dispersed points. Among such N_{e} points, it is necessary to measure the temperature of N points (it is adopted $N = 4$ in this paper) as well as the first point of the target only.

Estimating the K_i^j and q_{e}^j by the utilization of the linear least-squares error method of the inverse method, namely

$$\mathbf{T} = \mathbf{A}^{-1}\mathbf{D}\mathbf{C} = \mathbf{E}\mathbf{C}, \quad \mathbf{E} = \mathbf{A}^{-1}\mathbf{D} \tag{40}$$

$$\mathbf{C}_{\mathrm{est}} = (\mathbf{E}^{\mathrm{T}}\mathbf{E})^{-1}\mathbf{E}^{\mathrm{T}}\mathbf{T}_{\mathrm{grey}} \tag{41}$$

wherein $\mathbf{T}_{\mathrm{grey}}$ indicates the measured temperature plus the temperature acquired by the grey prediction, and $\mathbf{C}_{\mathrm{est}}$ indicates the estimation for K_i^j and q_{e}^j.

The total number of the K_i^j acquired is $N_{\mathrm{e}} - 1$, i.e. $K_1^j, K_2^j, K_3^j, \ldots, K_{Ne-1}^j$, in addition, it needs to perform the rolling grey prediction in the positive$x$direction according to last four successive K_i^j $(K_{Ne-4}^j, K_{Ne-3}^j, K_{Ne-2}^j, K_{Ne-1}^j)$, they are taken as the original data sequence of grey prediction. As a simplification, the operation performed in this section will adopt the common mean operator, and take $\alpha_m = 0.5$, not the variable of α_m value, and finally it is available to acquire the estimation of $K_{Ne}^j, K_{Ne+1}^j, K_{Ne+2}^j, \ldots, K_{n-1}^j$. Moreover, it is able to obtain the thermal conductivity k_i^j of each dispersed point based on Eqs. (21) and (19), wherein $1 \leqslant i \leqslant 11$.

Herein the process of grey prediction of K_i^j is similar to the process of T_i^j, namely the physical variable T of Eqs. (33), (34) and (13) will be replaced by K, as follows:

$$\hat{K}^{(1)}(5) = \left(K^{(0)}(1) - \frac{b}{a}\right)\mathrm{e}^{-4a} + \frac{b}{a} \tag{42}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = (\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{K} \tag{43}$$

$$\mathbf{B} = \begin{bmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ -z^{(1)}(4) & 1 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} K^{(0)}(2) \\ K^{(0)}(3) \\ K^{(0)}(4) \end{bmatrix} \tag{44}$$

Again, the predicted value of the $\hat{K}^{(0)}(5)$ will be gained after performing the IAGO operation based on Eq. (3), namely $\hat{K}^{(0)}(5) = \hat{K}^{(1)}(5) - K^{(1)}(4)$.

Additionally, the consideration for the practical measurement error is indispensable. The practical measurement error of temperature for the target is represented as

$$T_{\mathrm{meas}} = T_{\mathrm{exact}}(1 + \omega\sigma) \tag{45}$$

wherein T_{meas} indicates the measured temperature, T_{exact} indicates the actual temperature, ω indicates the value of a random variable $(-1 \leqslant \omega \leqslant 1)$, σ indicates the standard deviation for the measurement error of temperature.

5. Results and discussion

Here we reveal four examples of thermal conductivity in different forms in order to identify the feasibility of the method used in this paper. The initial temperature $T(x, 0)$ and the ambient temperature T_∞ of such target were all zero, and the left side heat flux q_0 is a constant. Due to a constant convection heat transfer coefficients h outside the right boundary, there is the right side heat flux $q_n(t) = h[T(1,t) - T_\infty]$. Firstly, we obtain the temperature field of the target with the numerical method based on the supposed thermal conductivity $k(x, T)$ together with the above-mentioned initial condition as well as boundary condition, wherein the temperature of this temperature field will be taken as the measured temperature at the measuring point. Upon the numerical operation, the setting for the total length of the space is 1, and the interval is $\Delta x = 0.1$. The time interval is $\Delta t = 0.001$, and the time point for the measurement is $t = 0.5$.

Suppose the common initial condition and boundary condition of these four examples are

$$\begin{cases} q_0 = 50, \quad h = 1, \quad T_\infty = 0, \quad t \geqslant 0 \\ q_n(t) = h[T(1,t) - T_\infty], \quad t \geqslant 0 \\ T(x,0) = 0, \quad 0 \leqslant x \leqslant 1 \end{cases} \tag{46}$$

First of all, we would like to emphasize that the initial temperature $T(x, 0)$, right side boundary condition (h or q_n) and ambient temperature T_∞ are all just for reckoning the temperature field, while carrying on the grey prediction of temperature measurement as well as the inverse method of thermal conductivity, it is not necessary to utilize these data at all. As long as there is an occurrence of left side boundary condition (heat flux q_0) together with the measured temperature, it is applicable to estimate both the thermal conductivity $k(x, T)$ and the heat flux q_{e} (or q_n) with the inverse method.

The discrepancy of each example only lies in the difference in thermal conductivity $k(x, T)$, and we suppose that the function of each thermal conductivity is as follows:

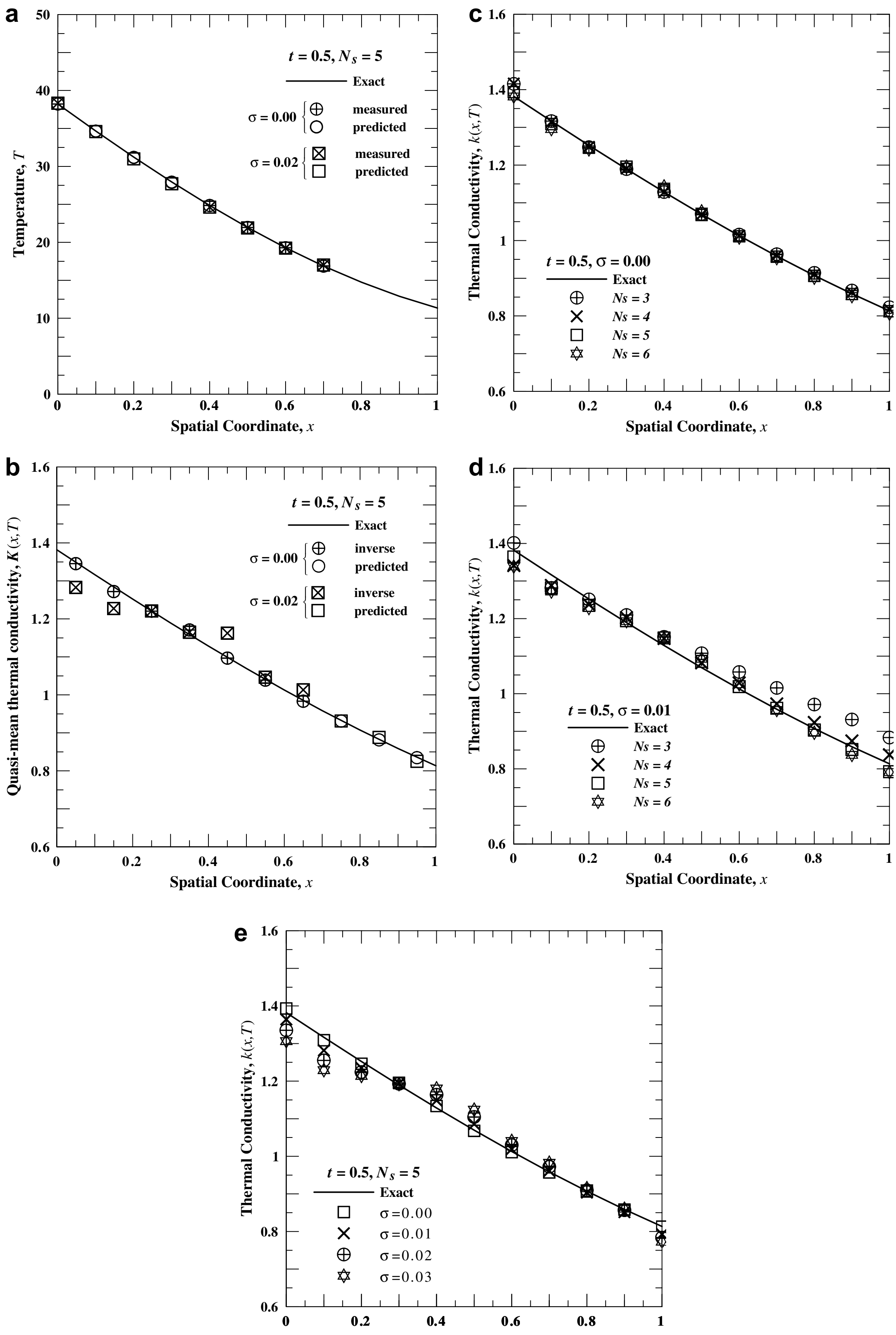

Fig. 2. Example 1 ($k(x,T) = 1 - 0.3x + 0.01T$): (a) The view of the measured and predicted temperature under the measurement error $\sigma = 0.00$ and $\sigma = 0.02$. (b) The view of the inverse and predicted "quasi-mean thermal conductivity" under the measurement error $\sigma = 0.00$ and $\sigma = 0.02$. (c) The comparison between the estimation of thermal conductivity and actual value, while $\sigma = 0.00$. (d) Narration like (c), but $\sigma = 0.01$. (e) The estimation of thermal conductivity under the condition with various measurement error (σ).

Example 1. The thermal conductivity is the function of position as well as temperature

$$k(x,T) = 1 - 0.3x + 0.01T \tag{47}$$

The position of measuring points for the temperature and grey prediction point are shown in Fig. 1. The predicted value of temperature (T) and "quasi-mean thermal conductivity" (K) are shown in Fig. 2a and b. The estimated thermal conductivity (k) are shown in Fig. 2c–e. While the relevant operation without including the measurement errors ($\sigma = 0.0$), the result will be very satisfactory, as shown in Fig. 2c. While the relevant operation includes the measurement errors ($\sigma = 0.01$), although the estimated value error is enlarged, the result is fair enough, besides, a different position for the initial point of the measuring segment N_s will affect the accuracy of the estimating value, as shown in Fig. 2d. With reference to Fig. 2d, it is found when $N_s = 4$ or $N_s = 5$, its accuracy is higher, due to the practical measuring segment that is located at the center section of such a target, therefore the number of grey prediction points at the left side or right side is very close and, thus the number of predicted points which need to be proceeded with the rolling grey prediction is getting less, so that the error value accumulated upon the operation of grey prediction will be decreased. Here, the observation is performed with $N_s = 5$, as shown in Fig. 2e, while comparing each other under the condition with various measurement errors, it is shown that all the solutions acquired are rather close to the actual values. The error values are shown in Table 1. With reference to the table, it is found that the difference between the estimated value and the actual value is very small; therefore the method used in this paper is reliable.

Example 2. Thermal conductivity indicates a quadratic function, as well as contains the positional and temperature parameter at the same time

$$k(x,T) = \frac{(x^2 - 3x + 3)}{2} + 0.01T \tag{48}$$

Although $k(x,T)$ is the quadratic function of position, its result was very similar to Example 1, as shown in Fig. 3a–c. These have proved that the quadratic function is suitable for this method, too.

Example 3. Thermal conductivity indicates the exponential function, as well as contains the positional and temperature parameter at the same time

$$k(x,T) = 1 - \frac{\exp(0.3x)}{2} + 0.01T \tag{49}$$

If compared with Examples 1 and 2, the circumstance is the same, but with a different function type. Nevertheless, it also showed similar operation results, as shown in Fig. 4a–c. Show again, the exponential function is suitable for this method, too.

Example 4. Thermal conductivity is the function of position only

$$k(x,T) = 1 - 0.3x \tag{50}$$

It is explainable, according to this example, that Although the thermal conductivity is the linear function of position only, the estimation is still applicable while using the proposed method, as shown in Fig. 5a and b, wherein the initial point for the temperature measurement is assigned to $N_s = 5$, the result is rather satisfactory. Even though the different time points $t = 0.3$, $t = 0.5$ and $t = 0.7$, have been assigned upon the estimation including the measurement errors ($\sigma = 0.01$), the accuracy of the results are all nearly identical, that is to say the thermal conductivity k is simply the function of position, which has nothing to do with the temperature (time), as shown in Fig. 5c. While viewing from Examples 1–4, the general applicability of the method used in this paper is explainable.

It is noticeable, according to the above four examples, that the type of thermal conductivity is basically different from the type of the temperature distribution presented by the target. Although there is a big difference among the types of the thermal conductivity, due to the heat absorption as well as heat transfer, it is generated a similar smooth curve (strictly increasing or strictly decreasing) for the temperature distribution of such target. And thus, upon temperature measurement, the proceeding of the grey prediction is facilitated to emphasize the feasibility related to the application of grey prediction method on the inverse thermal conductivity.

The method, proposed in the present study, used for estimating the thermal conductivity whereas its accuracy

Table 1
The difference between the estimation of thermal conductivity and the actual value, while the measurement errors $\sigma = 0.0$ and $\sigma = 0.01$

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$k(x,T)$ exact	1.3822	1.3165	1.2522	1.1895	1.1285	1.0695	1.0129	0.9587	0.9074	0.8589	0.8135
$k(x,T)$ estimated ($\sigma = 0.00$)	1.3933	1.3088	1.2462	1.1957	1.1339	1.0678	1.0111	0.9572	0.9064	0.8581	0.8128
error	0.0111	0.0077	0.0060	0.0062	0.0054	0.0017	0.0018	0.0015	0.0010	0.0008	0.0007
err (%)	0.803	0.585	0.479	0.521	0.479	0.159	0.178	0.156	0.110	0.093	0.086
$k(x,T)$ estimated ($\sigma = 0.01$)	1.3642	1.2818	1.2350	1.1942	1.1485	1.0860	1.0189	0.9613	0.9035	0.8514	0.7925
error	0.0180	0.0347	0.0172	0.0047	0.0200	0.0165	0.0060	0.0026	0.0039	0.0075	0.0210
err (%)	1.302	2.636	1.374	0.395	1.772	1.543	0.592	0.271	0.430	0.873	2.581

$k(x,T) = 1 - 0.3x + 0.01T$ (Example 1); $t = 0.5$, $N = 4$, $N_s = 5$, $N_e = 8$; error = |estimated − exact|, err = (error/exact) × 100.

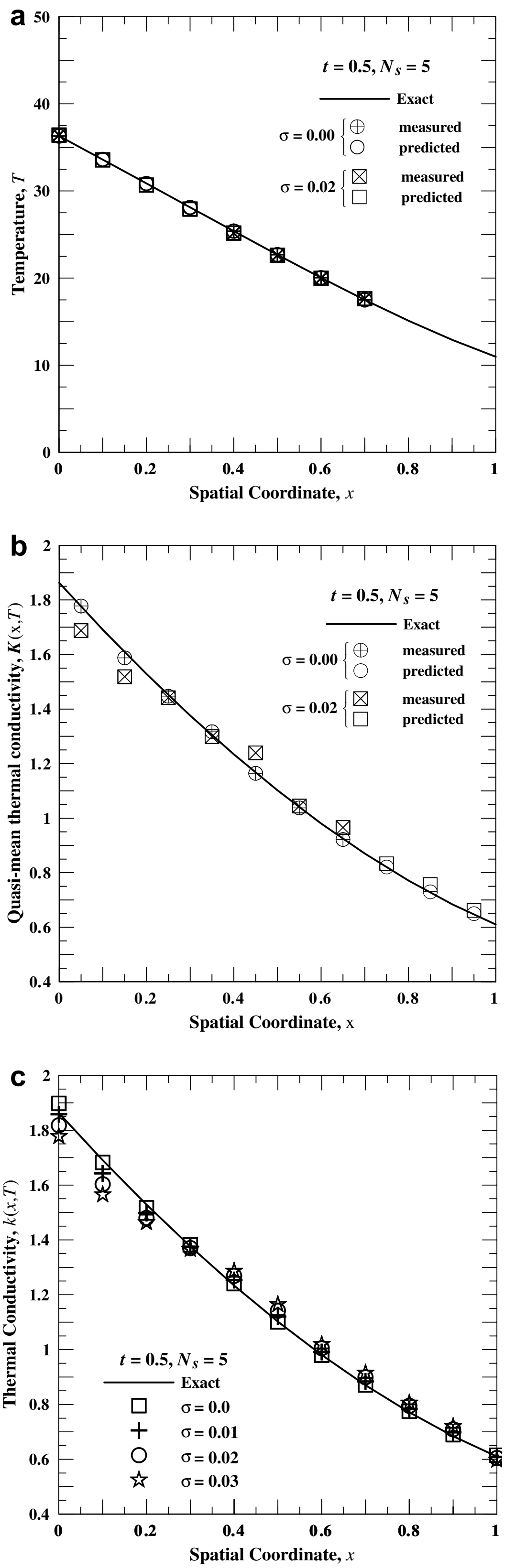

Fig. 3. Example 2 ($k(x,T) = 0.5(x^2 - 3x + 3) + 0.01T$): (a) The view of the measured and predicted temperature under the measurement error $\sigma = 0.00$ and $\sigma = 0.02$. (b) The view of the inverse and predicted ‘‘quasi-mean thermal conductivity’’ under the measurement error $\sigma = 0.00$ and $\sigma = 0.02$. (c) The estimation of thermal conductivity under the condition with various measurement error (σ).

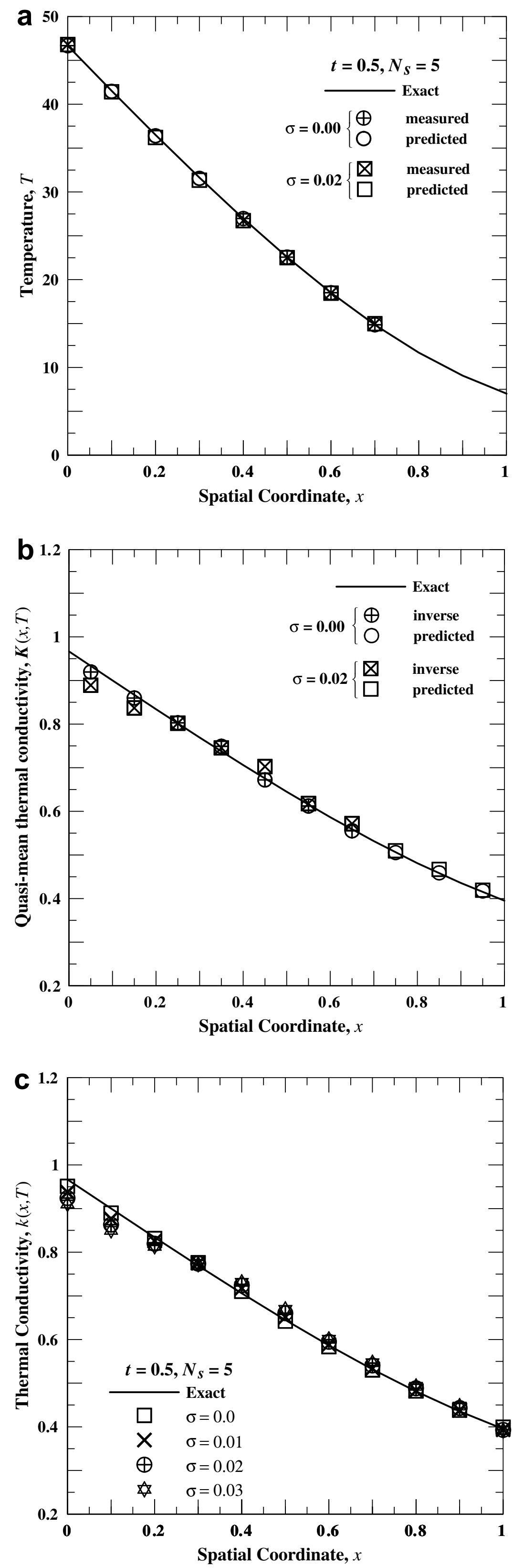

Fig. 4. Example 3 ($k(x,T) = 1 - 0.5e^{0.3x} + 0.01T$): (a) The view of the measured and predicted temperature under the measurement error $\sigma = 0.00$ and $\sigma = 0.02$. (b) The view of the inverse and predicted ‘‘quasi-mean thermal conductivity’’ under the measurement error $\sigma = 0.00$ and $\sigma = 0.02$. (c) The estimation of thermal conductivity under the condition with various measurement error (σ).

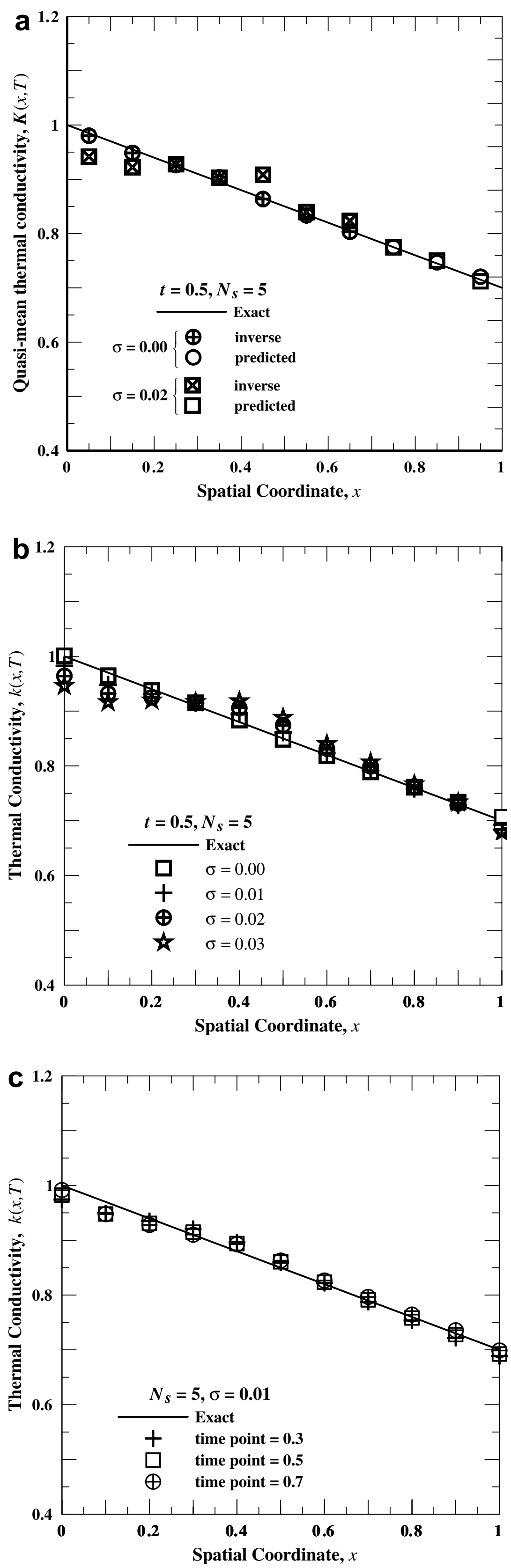

Fig. 5. Example 3 ($k(x,T) = 1 - 0.3x$): (a) The view of the inverse and predicted "quasi-mean thermal conductivity" under the measurement error $\sigma = 0.00$ and $\sigma = 0.02$. (b) The estimation of thermal conductivity under the condition with various measurement error. (c) The estimation of thermal conductivity measured at various time points.

is affected by both the measurement errors and the grey prediction errors, in which, the grey prediction can be partitioned into the grey prediction on the temperature of the former segment and the grey prediction on the quasi-mean thermal conductivity K of the latter segment. Among them, the errors in the grey prediction on the temperature can be corrected by means of improved rolling grey prediction (to correct the α_m value by the comparison of the measured temperature at the first point of the target), and the common mean operation is utilized in proceeding the grey prediction for quasi-mean thermal conductivity K, (i.e. $\alpha_m = 0.5$). This has been proved, even if the errors of measurement are indispensable. It is still eligible to measure temperature of N successive points (in this paper $N = 4$) in which the initial point of the measurement is the Fifth (or fourth) of the target (i.e. $N_s = 5$ or $N_s = 4$), in order to proceed the grey prediction, and then the following inverse operation will have an excellent result as well.

Acknowledgement

This research was supported by the National Science Council of Taiwan under Contract No. NSC 94-2212-E-006-019.

References

[1] C.Y. Yang, C.K. Chen, The boundary estimation in two-dimensional inverse heat conduction problems, J. Phys. D: Appl. Phys. 29 (2) (1996) 333–339.
[2] C.Y. Yang, A linear inverse model for the temperature-dependent thermal conductivity determination in one-dimensional problems, Appl. Math. Modell. 22 (1998) 1–9.
[3] J.H. Lin, C.K. Chen, Y.T. Yang, Inverse method for estimating thermal conductivity in one-dimensional heat conduction problems, J. Thermophys. Heat Transfer 15 (1) (2001) 34–41.
[4] J. Deng, Introduction of grey system, J. Grey Syst. 1 (1) (1989) 1–24.
[5] J. Deng, G. Hong, Grey Prediction: Theory and Applications, CHWA Publishing, Taipei, 1996, pp. 1.40–1.44.
[6] S.C. Chang, H.C. Yu, A variable P value rolling Grey forecasting model for Taiwan semiconductor industry production, Technol. Forecast. Social Change 72 (5) (2005) 623–640.
[7] Chu-lung Deng, The Basic Method of Grey System, Hwa Chung University of Science & Technology, 1996.
[8] Chih-yu Hsin, Yu-pin Tsai, The study on α value adjustment method of Grey prediction $Z(1)(k)$, in: Grey System Theory Seminar in 2001, Taiwan, 2001, pp. 305–308.